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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 96/80 APRIL

A.M. COHEN

A NEAR OCTAGON ASSOCIATED WITH HJ

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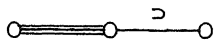
A near octagon associated with HJ

by

A.M. Cohen

ABSTRACT

A regular near octagon on 315 points with 3 points per line and 5 lines through each point is constructed from the conjugacy class of involutions of the simple group HJ of order 604800 whose centralizers contain 2 - Sylow subgroups. The relation with Buekenhout's diagram  is exhibited.

KEYWORDS & PHRASES: *generalized polygon, finite groups, quaternions*

1. INTRODUCTION AND DEFINITIONS

A *linear incidence system* is a system $(\mathcal{P},\mathcal{L})$ of points and lines such that every pair of points is on at most one line. The *point graph* $\Gamma(\mathcal{P},\mathcal{L})$ of such a system has $\mathcal{P}$ for its vertex set and pairs of collinear points for its edges. Two points $P,Q \in \mathcal{P}$ are said to have *distance* $d(P,Q) = d$ whenever their distance within the graph $\Gamma(\mathcal{P},\mathcal{L})$ is d.

A *near* 2m-*gon* as introduced in [SY] is by definition a linear incidence system $(\mathcal{P},\mathcal{L})$ such that

(i) For any point $P \in \mathcal{P}$ and line $\ell \in \mathcal{L}$, there is a unique point Q on ℓ with $\inf_{R\in\ell} d(R,P) = d(Q,P)$.

(ii) Every point lies on at least one line.

(iii) The distance between any two points is at most m.

Moreover, a near 2m-gon is said to have *order* (s,t) if each line contains 1+s points and each point contains 1+t lines, and it is called *regular* if for each pair of points P and Q at distance $d(P,Q) = d$, there are $1+t_d$ lines through P bearing a point at distance d-1 from Q.

This note is concerned with a proof of the following result.

<u>THEOREM</u>. *Let $(\mathcal{P},\mathcal{L})$ be the incidence system whose point set $\mathcal{P}$ is the conjugacy class of involutions of* HJ *(the Hall-Janko group) central in a 2-Sylow subgroup of* HJ *and whose line set $\mathcal{L}$ consists of the sets* {P,Q,R} *for* P,Q,R *three distinct pairwise commuting involutions from $\mathcal{P}$. Then $(\mathcal{P},\mathcal{L})$ is a regular near octagon.*

2. SKETCH OF PROOF

We shall employ the representation of HJ as a group generated by 315 homologies in the desarguean projective plane over the quaternions, cf. [C] or [T].

The skew field of quaternions $\mathbb{H}$ is the usual division algebra $\mathbb{H} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ whose multiplication is given by the following rules $ij = -ji = k$; $i^2 = j^2 = k^2 = -1$.
The quaternion conjugate $\bar{x} = x_0-x_1i-x_2j-x_3k$ of $x = x_0+x_1i+x_2j+x_3k \in \mathbb{H}$ leads to the multiplicative norm $N(x) = x\bar{x} \in \mathbb{R}_{\geq 0}$ for $x \in \mathbb{H}$. We think of $\mathbb{R}$ as

embedded in $\mathbb{H}$ by means of $r \longrightarrow r.1 \in \mathbb{H}$. We view $\mathbb{H}^3$ as a right vector space over $\mathbb{H}$ and $\mathbb{P}(\mathbb{H})$ as the associated projective plane, i.e. $\mathbb{P}(\mathbb{H})$ has points $x\mathbb{H}$ for $x \in \mathbb{H}^3\backslash\{0\}$ and lines $\{x\mathbb{H}+y\mathbb{H} \mid x\mathbb{H} \neq y\mathbb{H};\ x,y \in \mathbb{H}^3\backslash\{0\}\}$.

We introduce the following notations

$$\zeta = \frac{-1-i-j-k}{2} \quad ;$$

$$\tau = \frac{(1+\sqrt{5})}{2} \quad ;$$

$$Q = \{\pm 1, \pm i, \pm j, \pm k\} \ ;$$

$$(x|y) = \sum_{r=1}^{3} \bar{x}_r y_r \text{ whenever } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{H} ;$$

$$\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ;$$

$$\mathcal{P}_0 = \left\{ \begin{matrix} \varepsilon_u\mathbb{H},\ (\varepsilon_u+q\varepsilon_v)\mathbb{H},\ (p\varepsilon_u+q(\zeta^2+\tau)\varepsilon_v+r\varepsilon_w)\mathbb{H}, \\ (p\varepsilon_u+q\zeta(1-\tau)\varepsilon_v+r\zeta^2\tau\varepsilon_w)\mathbb{H} \end{matrix} \;\middle|\; \begin{matrix} p,q,r \in Q \\ pqr = \pm 1 \\ \{u,v,w\} = \{1,2,3\} \end{matrix} \right\} ;$$

$$\measuredangle(x\mathbb{H}, y\mathbb{H}) := \sqrt{\frac{(x|y)(y|x)}{(x|x)(y|y)}} \text{ for } x,y \in \mathbb{H}^3\backslash\{0\};$$

$$\mathcal{L}_0 = \{\{P,Q,R\} \subseteq \mathcal{P}_0 \mid \measuredangle(P,Q) = \measuredangle(P,R) = \measuredangle(Q,R) = 0\};$$

The *type* of $P \in \mathcal{P}_0$ (denoted by type (P)) is the unordered triple

$$\{\measuredangle(\varepsilon_1\mathbb{H}, P),\ \measuredangle(\varepsilon_2\mathbb{H}, P),\ \measuredangle(\varepsilon_3\mathbb{H}, P)\}.$$

For $a \in \mathbb{H}^3\backslash\{0\}$, we write r_a for the linear isometry on $\mathbb{H}^3$ defined by

$$r_a(x) = x-2a(a|a)^{-1}(a|x) \qquad (x \in \mathbb{H}^3)$$

and s_a for the corresponding involutorial homology of $\mathbb{P}(\mathbb{H})$.

CLAIMS.

(i) $\langle r_a | a\mathbb{H} \in \mathcal{P}_0 \rangle = \widetilde{HJ}$, the double cover of HJ, (see [G] or [T]).

(ii) $\{s_a | a\mathbb{H} \in \mathcal{P}_0\}$ is the class $\mathcal{P}$ of involutions of HJ described in the theorem.

(iii) For $a\mathbb{H}$, $b\mathbb{H} \in \mathcal{P}_0$ the homologies s_a, s_b commute if and only if $(a|b) = 0$.

(iv) The correspondence $a\mathbb{H} \leftrightarrow s_a$ for $a\mathbb{H} \in \mathcal{P}_0$ establishes an isomorphism between the linear incidence systems $(\mathcal{P}_0, \mathcal{L}_0)$ and $(\mathcal{P}, \mathcal{L})$.

(v) HJ is transitive on the flags $\{(P,\ell) | P \in \mathcal{P}_0, \ell \in \mathcal{L}_0, P \in \ell\}$.

(vi) The types of points in $\mathcal{P}_0$ are distributed according to the following table

type	# points of $\mathcal{P}_0$ with given type
$\{4,0,0\}$	3
$\{2,2,0\}$	24
$\{2,1,1\}$	192
$\{1,1+\tau,2-r\}$	96

(vii) Let $P,Q \in \mathcal{P}_0$. Then

$$d(P,Q) = \begin{cases} 0 \Leftrightarrow P = Q \\ 1 \Leftrightarrow \sphericalangle(P,Q) = 0 \\ 2 \Leftrightarrow \sphericalangle(P,Q) = 1/\sqrt{2} \\ 3 \Leftrightarrow \sphericalangle(P,Q) = 1/2 \\ 4 \Leftrightarrow \sphericalangle(P,Q) = \frac{\tau}{2}, \frac{\tau-1}{2} \end{cases}$$

(viii) For $P \in \mathcal{P}_0$, $\ell \in \mathcal{L}_0$ there is a unique $Q \in \mathcal{L}_0$ such that $d(P,Q)$ is minimal.

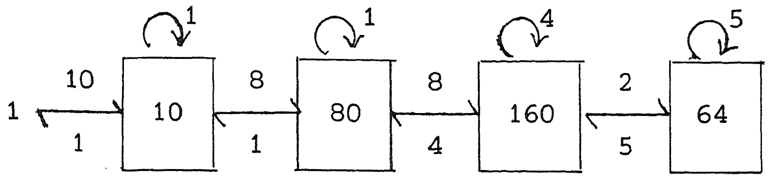
(ix) $(\mathcal{P}_0, \mathcal{L}_0)$ is a near 8-gon of order (2,4).

(x) There are $1+t_d$ lines through $\varepsilon_1\mathbb{H}$ bearing a point at distance $d-1$ from a point at distance d from $\varepsilon_1\mathbb{H}$, where t_d is as in the table

d	1	2	3	4
t_d	0	0	3	4

In particular (P_0, L_0) is regular. The above claims suffice for a proof of the theorem. We end this section by presenting parameters of the associated association scheme on P_0.

(xi) The diagram for the near octagon (P_0, L_0) is



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notation as in [SY].

(xii) Let A_i be the (0,1)-matrix with row and columns indexed by the points of P representing the relation $(x,y) \in A_i$ iff $d(x,y) = i$, then the algebra generated by the A_i is an association scheme of rank 4 with multiplication given by

$$A_iA_j = \sum_{h=0} P^k_{ij} A_k,$$

where

$$P_i = (P^k_{ij})_{0 \le k, j \le 4} \qquad \text{(rows indexed by k)}$$

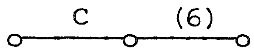
is as follows: $\qquad P_0 = I$

$$P_1 = \begin{pmatrix} 0 & 10 & 0 & 0 & 0 \\ 1 & 1 & 8 & 0 & 0 \\ 0 & 1 & 1 & 8 & 0 \\ 0 & 0 & 4 & 4 & 2 \\ 0 & 0 & 0 & 5 & 5 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 80 & 0 & 0 \\ 0 & 8 & 8 & 64 & 0 \\ 1 & 1 & 30 & 32 & 16 \\ 0 & 4 & 16 & 44 & 16 \\ 0 & 0 & 20 & 40 & 20 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 160 & 0 \\ 0 & 0 & 64 & 64 & 32 \\ 0 & 8 & 32 & 88 & 32 \\ 1 & 4 & 44 & 77 & 34 \\ 0 & 5 & 40 & 85 & 30 \end{pmatrix} \qquad P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 64 \\ 0 & 0 & 0 & 32 & 32 \\ 0 & 0 & 16 & 32 & 16 \\ 0 & 2 & 16 & 34 & 12 \\ 1 & 5 & 20 & 30 & 8 \end{pmatrix}$$

3. THE BUEKENHOUT DIAGRAM

In [B] the following diagram for HJ has been given.



The associated geometry can be derived from the near octagon by taking

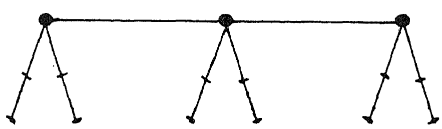
for 0-varieties the 100 hexagons of order (2,2) in $(\mathcal{P}_0,\mathcal{L}_0)$;

for 1-varieties the 3150 intersections of two hexagons that consist of 15 points and 7 lines;

for 2-varieties the 3150 3-claws in the near octagon consisting of a point, three lines through that point and the remaining 6 points on those lines;

for incidence between different varieties the inclusion relation.

Two distinct hexagons of order (2,2) intersect in either a generalized hexagon of order (2,1) or a 1-variety. Such a 1-variety is the point closure of a dual 3-claw:



However, not all subspaces of this form are intersections of two hexagons. As no intersection of two distinct generalized hexagons of order (2,1) within $(\mathcal{P}_0,\mathcal{L}_0)$ contains an ordinary hexagon, there is for each ordinary hexagon a unique generalized hexagon of order (2,1) containing it. This generalized hexagon is obtained as the socalled Cameron closure of the ordinary hexagon, i.e. as the smallest subspace X around this ordinary hexagon such that any point P of distance ≤ 1 to at least two distinct points in X is also contained in X. It follows that an ordinary hexagon lies in precisely 2 hexagons of order (2,2), so that the ordinary hexagon is not the proper replacement for the diamond in any conceivable analogue of "Yanushka's lemma" [SY, proposition (2.5)] for near 2n-gons whose minimal circuits have length 6.

4. REFERENCES

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